

FERMIONIC REALIZATION OF TOROIDAL LIE ALGEBRAS OF CLASSICAL TYPES

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ABSTRACT. We use fermionic operators to construct toroidal Lie algebras of classical types, including in particular that of symplectic affine algebras, which is first realized by fermions.

1. INTRODUCTION

Toroidal Lie algebras are natural generalization of the affine Kac-Moody algebras [MRY] that enjoy many similar interesting features. Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra of type X_n , and $R = \mathbb{C}[s, s^{-1}, t, t^{-1}]$ be the ring of Laurent polynomials in commuting variables s and t . By definition a 2-toroidal Lie algebra of type X_n is a perfect central extension of the iterated loop algebra $\mathfrak{g} \otimes R$.

Let Ω_R/dR be the Kähler differentials of R modulo the exact forms. The universal central extension of the iterated loop algebra is given by $T(X_n) = (\mathfrak{g} \otimes R) \oplus \Omega_R/dR$. Any 2-toroidal Lie algebra of type X_n is a homomorphic image of this toroidal Lie algebra. The center of $T(X_n)$ is Ω_R/dR , which is an infinite-dimensional vector space. The Laurent polynomial ring R induces a natural \mathbb{Z}^2 -gradation on $T(X_n)$. The center is given by $\Omega_R/dR = \bigoplus_{\sigma \in \mathbb{Z}^2} \mathcal{Z}(g)_\sigma$, with $\dim \mathcal{Z}_\sigma = 1$ if $\sigma \neq (0, 0)$, and 2 if $\sigma = (0, 0)$. We denote by c_0 and c_1 the two standard degree zero central elements in the toroidal Lie algebra $T(X_n)$. A module of $T(X_n)$ is called a level- (k_0, k_1) module if the standard pair of central elements (c_0, c_1) acts as (k_0, k_1) for some complex numbers k_0 and k_1 . In this work we will only consider modules with $k_0 \neq 0$.

Representation theory of toroidal Lie algebras have been studied extensively in recent years (for example see: [MRY] [BB], [EM], [FM], [T1], [T2], [JMT], [XH]). Some exceptional types were studied in [LH]. Most of these realizations are bosonic. In [FF] bosonic and fermionic constructions for the classical affine Lie algebras are given. Motivated

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by [FF], Gao [G] gave both bosonic and fermionic constructions of the extended affine general linear algebras. In [L], Lau gave more general bosonic and fermionic constructions which included as special cases the constructions in [FF] and [G] for the affine and extended affine general linear algebras as well as Virasoro algebra and W -algebra.

In this paper we extend the fermionic construction of Feingold-Frenkel [FF] to give a unified fermionic construction for all 2-toroidal Lie algebras of classical types. In particular, it includes the special case of $q = 1$ in [G]. Our main idea is to construct the operators corresponding to special nodes in the affine Dynkin diagrams. It turns out that the idea of ghost fields plays an important role in our construction. By introducing special auxiliary fields we are able to define actions for the root vectors corresponding to the special nodes.

Another new feature of our construction is that we have succeeded in realizing the type C toroidal Lie algebras exclusively using fermions, which contains the affine symplectic Lie algebras as a special case. In the original Feingold-Frenkel construction the type C case was not available in fermionic construction, and it has been an open problem for nearly 25 years. This is because one needs to use quadratic expressions to represent all root vectors in the algebra. If $b(z)$ is a fermionic field, then $: b(z)b(z) := 0$, thus it is impossible to directly use fermionic operators to realize the special node $\alpha_n = 2\varepsilon_n$ in type C toroidal Lie algebras, and we have found new ways to realize the symplectic affine Lie algebras by introducing new fermions. In a subsequent paper we will generalize this idea to the bosonic picture [JMX].

The structure of this paper is as follows. In section 2 we define the toroidal Lie algebra, and state MRY-presentation [MRY] of the toroidal algebra in terms of generators and relations. In section 3 we start with a finite rank lattice with a symmetric bilinear form and define a Fock space and some vertex operators, which in turn give level (1,0) representations of the toroidal Lie algebra of type A_n, B_n, C_n or D_n . The proof is an extensive analysis of the operator product expansions for the field operators. We also include the verification of the Serre relations.

2. TOROIDAL LIE ALGEBRAS

Let \mathfrak{g} be the complex simple Lie algebra over \mathbb{C} of type X_n , and $R = \mathbb{C}[s, s^{-1}, t, t^{-1}]$ be the ring of Laurent polynomials in commuting variables. We consider the iterated loop algebra $\mathfrak{g}(R) = \mathfrak{g} \otimes R$. A toroidal Lie algebra of type X_n is a perfect central extension of the iterated loop algebra $\mathfrak{g}(R)$. Let Ω_R be the R -module of differentials

with differential mapping $d : R \rightarrow \Omega_R$, such that $d(g_1g_2) = (dg_1)g_2 + g_1(dg_2)$ for all g_1, g_2 in R . Let $- : \Omega_R \rightarrow \Omega_R/dR$ be the canonical linear map for which $\overline{dg} = 0$ for all $g \in R$. Define the vector space

$$T(X_n) := (\mathfrak{g} \otimes R) \oplus \Omega_R/dR$$

with the following bracket operation defined by

$$[x \otimes g_1, y \otimes g_2] = [x, y] \otimes g_1g_2 + (x, y)\overline{g_2dg_1},$$

and Ω_R/dR central, for $x, y \in \mathfrak{g}$, $g_1, g_2 \in R$, where (\cdot, \cdot) is the trace form. From [MRY] we know that $T(X_n)$ is a perfect Lie algebra and is the universal central extension of the iterated loop algebra $\mathfrak{g}(R)$. Therefore, any toroidal Lie algebra of type X_n is a homomorphic image of $T(X_n)$. The gradation of the polynomial ring R gives a natural \mathbb{Z}^2 -gradation to the toroidal Lie algebra $T(X_n) := \bigoplus_{\sigma \in \mathbb{Z}^2} T(X_n)_\sigma$, where $T(X_n)_\sigma$ is spanned by $x \otimes s^{m_0}t^{m_1}$, $\overline{s^{m_0}t^{m_1}s^{-1}ds}$ and $\overline{s^{m_0}t^{m_1}t^{-1}dt}$ for $\sigma = (m_0, m_1) \in \mathbb{Z}^2$ and $x \in \mathfrak{g}$. The condition $\overline{dg} = 0$ for all $g \in R$ implies that $\overline{m_0s^{m_0}t^{m_1}s^{-1}ds} + \overline{m_1s^{m_0}t^{m_1}t^{-1}dt} = 0$ for all $m_0, m_1 \in \mathbb{Z}$. Therefore $\dim T(X_n)_\sigma = 1 + \dim(\mathfrak{g})$ if $\sigma \neq (0, 0)$, and $2 + \dim(\mathfrak{g})$ if $\sigma = (0, 0)$. In particular, $T(X_n)_{(0,0)}$ is spanned by $x \otimes 1$ for $x \in \mathfrak{g}$, and central elements $\overline{s^{-1}ds}$, $\overline{t^{-1}dt}$. We denote these two degree zero central elements by c_0 and c_1 .

The most interesting quotient algebra of the toroidal Lie algebra $T(X_n)$ is the double affine algebra, denoted by $T_0(X_n)$, that is the toroidal Lie algebra of type X_n with a two dimensional center. The double affine algebra is the quotient of $T(X_n)$ modulo all the central elements with degree other than zero. In fact, $T_0(X_n)$ has the following realization

$$T_0(X_n) = (\mathfrak{g} \otimes R) \oplus \mathbb{C}c_0 \oplus \mathbb{C}c_1$$

with the Lie product

$$[x \otimes g_1, y \otimes g_2] = [x, y] \otimes g_1g_2 + \Phi(g_2\partial_s g_1)c_0 + \Phi(g_2\partial_t g_1)c_1$$

for all $x, y \in \mathfrak{g}$, $g_1, g_2 \in R$, where Φ is a linear functional on R defined by $\Phi(s^k t^m) = 0$ if $(k, m) \neq (0, 0)$ and $\Phi(s^k t^m) = 1$, if $(k, m) = (0, 0)$ for all $k, m \in \mathbb{Z}$.

Definition 2.1 *If M is a module for a toroidal Lie algebra of type X_n , we call M a level- (k_0, k_1) module for some complex numbers k_0, k_1 if the degree zero central elements c_0, c_1 act on M as constants k_0, k_1 respectively.*

In the present paper we want to give concrete construction of level- $(1, 0)$ module for the toroidal Lie algebra $T(X_n)$, and also for the double affine algebra $T_0(X_n)$, for $X = A, B, C, D$.

Let $(a_{ij})_{n+1 \times n+1}$ be the generalized Cartan matrix of the affine algebra $X_n^{(1)}$, and $Q := \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$ its root lattice. The toroidal Lie algebra $T(X_n)$ has the following presentation [MRY] with generators $\not\ell$, $\alpha_i(k)$, and $x_k(\pm\alpha_i)$ for $i = 0, 1, \dots, n$, $k \in \mathbb{Z}$, and the following relations:

- (R0). $[\not\ell, \alpha_i(k)] = 0 = [\not\ell, x_k(\pm\alpha_i)];$
- (R1). $[\alpha_i(k), \alpha_j(m)] = k(\alpha_i|\alpha_j)\delta_{k+m,0}\not\ell;$
- (R2). $[\alpha_i(k), x_m(\pm\alpha_j)] = \pm(\alpha_i|\alpha_j)x_{k+m}(\pm\alpha_j);$
- (R3). $[x_k(\alpha_i), x_m(-\alpha_j)] = -\delta_{ij}\frac{2}{(\alpha_i|\alpha_j)}\{\alpha_i(k+m) + k\delta_{k+m,0}\not\ell\};$
- (R4). $[x_k(\alpha_i), x_m(\alpha_i)] = 0 = [x_k(-\alpha_i), x_m(-\alpha_i)];$
 $(adx_0(\alpha_i))^{-a_{ij}+1}x_m(\alpha_j) = 0, \text{ if } i \neq j;$
 $(adx_0(-\alpha_i))^{-a_{ij}+1}x_m(-\alpha_j) = 0, \text{ if } i \neq j;$

for $i, j = 0, 1, \dots, n$ and $k, m \in \mathbb{Z}$. It is known [MRY] that there is an isomorphism ψ between the two presentations of $T(X_n)$. In this paper we will identify the two presentations of the toroidal Lie algebra $T(X_n)$ via this isomorphism ψ . In particular, under this isomorphism we can identify the degree zero central elements $c_0 = \not\ell$ and $c_1 = \delta(0)$, where δ is the null root in Q . We also remark that in our definition of the toroidal Lie algebra, we use the root operators $\alpha_i(k)$ instead of coroot operators $\alpha_i^\vee(k)$ as in [MRY].

Following [MRY], we introduce a $\mathbb{Z} \times Q$ -gradation on $T(X_n)$ by assigning $\deg \not\ell = (0, 0)$, $\deg \alpha_i(k) = (k, 0)$, $\deg x_k(\pm\alpha_i) = (k, \pm\alpha_i)$, with $i = 0, 1, \dots, n$ and $k \in \mathbb{Z}$. We denote by T_k^α the subspace of $T(X_n)$ spanned by the elements with degree (k, α) for $k \in \mathbb{Z}$, $\alpha \in Q$. Then, under the isomorphism ψ , we have $\psi^{-1}(\overline{s^k t^{-1} dt}) = \delta(k) \in T_k^0$, and $\psi^{-1}(\overline{s^k t^r s^{-1} ds}) \in T_k^{r\delta}$.

Let z, w, z_1, z_2, \dots be formal variables. We define formal power series with coefficients from the toroidal Lie algebra $T(X_n)$:

$$\alpha_i(z) = \sum_{n \in \mathbb{Z}} \alpha_i(n) z^{-n-1}, \quad x(\pm\alpha_i, z) = \sum_{n \in \mathbb{Z}} x_n(\pm\alpha_i) z^{-n-1},$$

for $i = 0, 1, \dots, n$. We will use the delta function

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} w^n z^{-n-1}$$

Using $\frac{1}{z-w} = \sum_{n=0}^{\infty} z^{-n-1}w^n$, $|z| > |w|$, we have the following useful expansions:

$$\begin{aligned}\delta(z-w) &= \iota_{z,w}((z-w)^{-1}) + \iota_{w,z}((w-z)^{-1}), \\ \partial_w \delta(z-w) &= \iota_{z,w}((z-w)^{-2}) - \iota_{w,z}((w-z)^{-2}),\end{aligned}$$

where $\iota_{z,w}$ means expansion when $|z| > |w|$. For simplicity in the following we will drop $\iota_{z,w}$ if it is clear from the context.

Now the Lie algebra structure of $T(X_n)$ can be expressed in terms of the following power series identities:

$$\begin{aligned}(R0)' [\not\epsilon, \alpha_i(z)] &= 0 = [\not\epsilon, x(\pm\alpha_i, z)]; \\ (R1)' [\alpha_i(z), \alpha_j(w)] &= (\alpha_i|\alpha_j)\partial_w \delta(z-w) \not\epsilon; \\ (R2)' [\alpha_i(z), x(\pm\alpha_j, w)] &= \pm(\alpha_i|\alpha_j)x(\pm\alpha_j, w)\delta(z-w); \\ (R3)' [x(\alpha_i, z), x(-\alpha_j, w)] &= -\delta_{ij} \frac{2}{(\alpha_i|\alpha_j)} \{ \alpha_i(w)\delta(z-w) + \partial_w \delta(z-w) \not\epsilon \}; \\ (R4)' [x(\alpha_i, z), x(\alpha_i, w)] &= 0 = [x(-\alpha_i, z), x(-\alpha_i, w)],\end{aligned}$$

and for $0 \leq i \neq j \leq n$,

$$\begin{aligned}\text{adx}(\pm\alpha_i, z_1)x(\pm\alpha_j, z_2) &= 0, \text{ if } a_{ij} = 0 \\ (\text{adx}(\pm\alpha_i, z_1))(\text{adx}(\pm\alpha_i, z_2))x(\pm\alpha_j, z_3) &= 0, \text{ if } a_{ij} = -1 \\ (\text{adx}(\pm\alpha_i, z_1))(\text{adx}(\pm\alpha_i, z_2))(\text{adx}(\pm\alpha_i, z_3))x(\pm\alpha_j, z_4) &= 0, \\ &\text{if } a_{ij} = -2.\end{aligned}$$

3. REPRESENTATIONS OF THE TOROIDAL ALGEBRA

In this section we give a fermionic realizations for the toroidal Lie algebra of classical types A_{n-1} , B_n , D_n and C_n .

Let ε_i ($i = 0, \dots, n+1$) be a set of orthonormal basis of the vector space \mathbb{C}^{n+2} equipped with the inner product $(\cdot | \cdot)$ such that

$$(\varepsilon_i | \varepsilon_j) = \delta_{ij},$$

Let $P_0 = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_n$ be a sublattice of rank n , and let $\bar{c} = \frac{1}{\sqrt{2}}(\varepsilon_0 + i\varepsilon_{n+1})$ correspond to the null vector δ and $\bar{d} = \frac{1}{\sqrt{2}}(\varepsilon_0 - i\varepsilon_{n+1})$ be the dual gradation operator. Then

$$(\bar{c} | \bar{c}) = (\bar{c} | \varepsilon_i) = 0$$

$$(\bar{d} | \bar{d}) = (\bar{d} | \varepsilon_i) = 0$$

$$(\bar{c} | \bar{d}) = 1$$

for $i = 1, \dots, n$.

The simple roots for the classical finite dimensional Lie algebras can be realized simply by defining the simple roots as follows:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n; \text{ for } A_{n-1}.$$

$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n$; for B_n .

$\alpha_1 = \frac{\varepsilon_1 - \varepsilon_2}{\sqrt{2}}, \dots, \alpha_{n-1} = \frac{\varepsilon_{n-1} - \varepsilon_n}{\sqrt{2}}, \alpha_n = \sqrt{2}\varepsilon_n$; for C_n .

$\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$; for D_n .

Then the set of positive roots are:

$$\Delta_+ = \begin{cases} \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n\}, & \text{Type } A_{n-1} \\ \{\varepsilon_i, \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\}, & \text{Type } B_n \\ \{\sqrt{2}\varepsilon_i, \frac{1}{\sqrt{2}}(\varepsilon_i \pm \varepsilon_j) | 1 \leq i < j \leq n\}, & \text{Type } C_n \\ \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\}, & \text{Type } D_n. \end{cases}$$

The highest (long) root α_{max} for each type is given as follows:

$$\alpha_{max} = \begin{cases} \varepsilon_1 - \varepsilon_n, & \text{Type } A_{n-1} \\ \sqrt{2}\varepsilon_1 & \text{Type } C_n \\ \varepsilon_1 + \varepsilon_2, & \text{Type } B_n \text{ or } D_n \end{cases}$$

We further introduce the element

$$\alpha_0 = \bar{c} - \alpha_{max}$$

in the lattice and then define $\beta = -\bar{c} + \varepsilon_1$ for type ABD , and $\beta = -\sqrt{2}\bar{c} + \varepsilon_1$ for type C . Then we have

$$\alpha_0 = \varepsilon_n - \beta \text{ for } A_{n-1}; \quad -\beta - \varepsilon_2 \text{ for } B_n, D_n; \quad \text{or } -\frac{1}{\sqrt{2}}(\beta + \varepsilon_1) \text{ for } C_n.$$

Note that $(\beta|\beta) = 1, (\beta|\varepsilon_i) = \delta_{1i}$.

Then $P = \mathbb{Z}\bar{c} \oplus \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_n$ and $Q = \mathbb{Z}\bar{c} \oplus \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_k = \mathbb{Z}\alpha_0 \oplus \dots \oplus \mathbb{Z}\alpha_k$ ($k = n-1$ for type A_{n-1} and $k = n$ for types B_n, D_n) are the weight lattice and root lattice for the corresponding affine Lie algebra, and $P_0 = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_k$ and $Q_0 = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_k$ ($k = n-1$ for type A_{n-1} and $k = n$ for types B_n, D_n) are the weight lattice and root lattice for the simple Lie algebras A_{n-1}, B_n, C_n, D_n . Then $(\alpha_i|\alpha_j) = d_i a_{ij}$, where a_{ij} are the entries of the affine Cartan matrix of type $(ABCD)^{(1)}$, and the d_i 's are given by:

$$(d_0, d_1, \dots, d_k) = \begin{cases} \{1, 1, \dots, 1, 1\}, & , k = n-1, \text{Type } A_{n-1} \\ \{1, 1, \dots, 1, \frac{1}{2}\}, & , k = n, \text{Type } B_n \\ \{1, \frac{1}{2}, \dots, \frac{1}{2}, 1\}, & k = n, \text{Type } C_n \\ \{1, 1, \dots, 1, 1\}, & , k = n, \text{Type } D_n. \end{cases}$$

We introduce infinite dimensional Clifford algebras as follows. We first let $\varepsilon_i, 1 \leq i \leq n$ be orthonormal vectors such that $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$ and $(\varepsilon_i|\varepsilon_j) = 0$. Let $\tilde{P}_{\mathbb{C}}$ be the \mathbb{C} -vector space spanned by \bar{c} and

$\varepsilon_i, 1 \leq i \leq n$ for types AD , by \bar{c} and $\varepsilon_i, \varepsilon_{\bar{i}}, 1 \leq i \leq n$ in type C , and by $\bar{c}, \varepsilon_i, 1 \leq i \leq n$ and a ghost element e (to be defined later) for types B . We also denote $\bar{\beta} = -\sqrt{2}\bar{c} + \varepsilon_{\bar{1}}$. Then we define $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, where both subspaces $\mathcal{C}_0 = \tilde{P}_{\mathbb{C}}$ and $\mathcal{C}_1 = \tilde{P}_{\mathbb{C}}^*$ are maximal isotropic subspaces. The symmetric bilinear form on \mathcal{C} is given by

$$\langle b^*, a \rangle = \langle a, b^* \rangle = (a|b), \quad \langle a, b \rangle = \langle a^*, b^* \rangle = 0, \quad a, b \in \tilde{P}_{\mathbb{C}}$$

In this way we have a maximal polarization of \mathcal{C} .

The Clifford algebra $Cl(\tilde{P})$ is generated by the central element C and the elements $a(k)$ and $a^*(k)$, where $a \in \tilde{P}_{\mathbb{C}} = \tilde{P} \otimes \mathbb{C}$, $a^* \in \tilde{P}_{\mathbb{C}}^*$, and $k \in \mathbb{Z} + 1/2$ subject to the relations:

$$\begin{aligned} \{a(k), b(l)\} &= 0 \\ \{a^*(k), b^*(l)\} &= 0 \\ \{a(k), b^*(l)\} &= (a|b)\delta_{k,-l}C \end{aligned}$$

where $a, b \in \tilde{P}_{\mathbb{C}}$. Note that the anticommutation relations can be simply written as

$$\{u(k), v(l)\} = \langle u, v \rangle \delta_{k,-l}C, \quad u, v \in \mathcal{C}.$$

The representation space is the infinite dimensional vector space

$$V = \bigotimes_{a_i} \left(\bigotimes_{k \in \mathbb{Z}_+ + 1/2} \mathbb{C}[a_i(-k)] \bigotimes_{k \in \mathbb{Z}_+ + 1/2} \mathbb{C}[a_i^*(-k)] \right)$$

where a_i runs through any basis in \tilde{P} , say \bar{c}, ε_i 's and $\varepsilon_{\bar{i}}$'s.

The Clifford algebra acts on the space V by the usual action: $a(-k)$ acts as a creation operator, $a(k)$ as an annihilation operator and C acts as the identity.

For any two fermionic fields $u(z) = \sum_{n \in \mathbb{Z} + 1/2} a(n)z^{-n-1/2}$ and $v(z) = \sum_{n \in \mathbb{Z} + 1/2} b(n)z^{-n-1/2}$ we define the normal ordering : $u(z)v(w)$: by their components:

$$(3.1) \quad : u(m)v(n) : = \begin{cases} u(m)v(n) & m < 0 \\ -v(n)u(m) & m > 0 \end{cases}.$$

It follows from the definition that the normal ordering satisfies the relation:

$$: u(z)v(w) : = - : v(w)u(z) : .$$

Based on the normal product of two fields, we can define the normal product of n fields inductively as follows. We define that

$$\begin{aligned} & : u_1(z_1)u_2(z_2) \cdots u_n(z_n) : \\ & =: u_1(z_1)(: u_2(z_2) \cdots u_n(z_n) :) :, \end{aligned}$$

and then use induction till we reach 2 fields.

We define the contraction of two states by

$$\underbrace{a(z)b(w)} = a(z)b(w) - : a(z)b(w) :,$$

which contains all poles for $a(z)b(w)$. In general, the contraction of several pairs of states is given inductively by the following rule.

Proposition 3.1. *The basic operator product expansions are: for $x, y \in \mathcal{C}$ we have*

$$\underbrace{u(z)v(w)} = \frac{\langle u, v \rangle}{z - w}$$

In particular we have for $a, b \in P_{\mathbb{C}}$

$$\begin{aligned} \underbrace{a(z)b(w)} &= \underbrace{a^*(z)b^*(w)} = 0, \\ \underbrace{a(z)b^*(w)} &= \underbrace{a^*(z)b(w)} = \frac{(a, b)}{z - w}, \end{aligned}$$

Proof. In fact one has

$$\begin{aligned} a(z)b^*(w) &=: a(z)b^*(w) : + \sum_{m \in \mathbb{Z}_+ + 1/2} [a(m), b^*(n)] z^{-m-1/2} w^{-n-1/2} \\ &=: a(z)b^*(w) : + (a, b) C \sum_{0 < m \in \mathbb{Z}_+ + 1/2} z^{-m-1/2} w^{-m-1/2} \\ &=: a(z)b^*(w) : + \frac{(a, b)}{z - w}. \end{aligned}$$

The other OPEs are proved in the same manner. \square

Proposition 3.2. *The fermionic fields satisfy the following anticommutation relations:*

$$\begin{aligned} \{a(z), b(w)\} &= \{a^*(z), b^*(w)\} = 0, \\ \{a(z), b^*(w)\} &= (a, b)\delta(z - w). \end{aligned}$$

Proof. As in the proof of Proposition 3.1, we have

$$\begin{aligned} \{u(z), v(w)\} &= u(z)v(w) + v(w)u(z) \\ &= \langle u, v \rangle \left(\frac{1}{z - w} + \frac{1}{w - z} \right) \\ &= \langle u, v \rangle \delta(z - w) \end{aligned}$$

since $: u(z)v(w) : + : v(w)u(z) := 0$. \square

The following result is easily obtained by Wick's theorem.

Proposition 3.3. *The brackets among normal order products are given by:*

$$\begin{aligned} [: r_1(z) r_2(z) :, : s_1(w) s_2(w) :] = \\ < r_1, s_2 > : r_2(z) s_1(z) : \delta(z-w) - < r_1, s_1 > : r_2(z) s_2(z) : \delta(z-w) \\ + < r_2, s_1 > : r_1(z) s_2(z) : \delta(z-w) - < r_2, s_2 > : r_1(z) s_1(z) : \delta(z-w) \\ + (< r_1, s_2 > < r_2, s_1 > - < r_1, s_1 > < r_2, s_2 >) \partial_w \delta(z-w). \end{aligned}$$

The inner product of the underlying Lie algebra can be extended to that of the linear factors as follows:

$$< : r_1 r_2 :, : s_1 s_2 : > = - < r_1, s_1 > < r_2, s_2 > + < r_1, s_2 > < r_2, s_1 >$$

For any root vector $\alpha = \sum_{i \in I} \varepsilon_i - \sum_{j \in I'} \varepsilon_j$, we define the field operator $X(\alpha, z)$ as follows.

$$(3.2) \quad X(\alpha, z) = : \prod_{i \in I}^{\rightarrow} \varepsilon_i(z) \prod_{j \in I'}^{\rightarrow} \varepsilon_j^*(z) :,$$

where $\prod_{i \in I}^{\rightarrow}$ means the ordered product according to the natural order in I .

Furthermore, we introduce a ghost field $e(z) = \sum_{k \in \mathbb{Z}+1/2} e(k) z^{-k-1/2}$ such that

$$\begin{aligned} (e|e) &= -1, (e|\varepsilon_i) = 0 \\ \{e(k), e(l)\} &= -\delta_{k,-l} \end{aligned}$$

Then we have

$$\begin{aligned} \underbrace{e(z) \varepsilon_i(w)} &= \underbrace{e(z) \varepsilon_i^*(w)} = 0 \\ \underbrace{e(z) e(w)} &= \frac{-1}{z-w} \end{aligned}$$

Theorem 3.1. *Under the following map we have a level (1,0) representation of the toroidal Lie algebra of classical types:*

$$\begin{aligned}
X(\alpha_0, z) &= \begin{cases} : \varepsilon_n(z) \beta^*(z) : & \text{for } A_{n-1} \\ \frac{1}{\sqrt{2}} (: \beta^*(z) \varepsilon_1^*(z) : - : \varepsilon_1^*(z) \bar{\beta}^*(z) :) & \text{for } C_n \\ : \beta^*(z) \varepsilon_2^*(z) : & \text{for } B_n \text{ or } D_n \end{cases} \\
X(-\alpha_0, z) &= \begin{cases} : \varepsilon_n^*(z) \beta(z) : & \text{for } A_{n-1} \\ \frac{1}{\sqrt{2}} (: \beta(z) \varepsilon_1(z) : - : \varepsilon_1(z) \bar{\beta}(z) :) & \text{for } C_n \\ : \beta(z) \varepsilon_2(z) : & \text{for } B_n \text{ or } D_n \end{cases} \\
X(\alpha_i, z) &= \begin{cases} : \varepsilon_i(z) \varepsilon_{i+1}^*(z) : & 1 \leq i \leq n-1, \text{ for } A_{n-1} B_n D_n \\ : \varepsilon_i(z) \varepsilon_{i+1}^*(z) : - : \varepsilon_{i+1}^*(z) \varepsilon_i^*(z) : & \text{for } C_n \end{cases} \\
X(-\alpha_i, z) &= \begin{cases} : \varepsilon_i^*(z) \varepsilon_{i+1}(z) : , & 1 \leq i \leq n-1, \text{ for } A_{n-1} B_n D_n \\ : \varepsilon_i^*(z) \varepsilon_{i+1}(z) : - : \varepsilon_{i+1}^*(z) \varepsilon_i(z) : & \text{for } C_n \end{cases} \\
X(\alpha_n, z) &= \begin{cases} \sqrt{2} : \varepsilon_n(z) e(z) : & \text{for } B_n \\ : \varepsilon_n(z) \varepsilon_n^*(z) : & \text{for } C_n \\ : \varepsilon_{n-1}(z) \varepsilon_n(z) : & \text{for } D_n \end{cases} \\
X(-\alpha_n, z) &= \begin{cases} \sqrt{2} : e(z) \varepsilon_n^*(z) : & \text{for } B_n \\ : \varepsilon_n^*(z) \varepsilon_n(z) : & \text{for } C_n \\ : \varepsilon_{n-1}^*(z) \varepsilon_n^*(z) : & \text{for } D_n \end{cases}
\end{aligned}$$

and the fields for the simple roots are represented by

$$\begin{aligned}
\alpha_0(z) &= \begin{cases} : \varepsilon_n(z) \varepsilon_n^*(z) : - : \beta(z) \beta^*(z) : & \text{for } A_{n-1} \\ : \beta^*(z) \beta(z) : + : \varepsilon_2^*(z) \varepsilon_2(z) : & \text{for } B_n \text{ or } D_n \\ \frac{1}{2} (: \beta^*(z) \beta(z) : + : \varepsilon_1^*(z) \varepsilon_1(z) : \\ \quad + : \bar{\beta}^*(z) \bar{\beta}(z) : + : \varepsilon_1^*(z) \varepsilon_1(z) :) & \text{for } C_n \end{cases} \\
\alpha_i(z) &= \begin{cases} : \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : & 1 \leq i \leq n-2, \\ \frac{1}{2} (: \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) : \\ \quad - : \varepsilon_i^*(z) \varepsilon_{i+1}(z) : + : \varepsilon_{i+1}^*(z) \varepsilon_i(z) :) & \text{for } C_n \end{cases} \\
\alpha_n(z) &= \begin{cases} : \varepsilon_n(z) \varepsilon_n^*(z) : & \text{for } B_n \\ : \varepsilon_n(z) \varepsilon_n^*(z) : - : \varepsilon_n(z) \varepsilon_n^*(z) : & \text{for } C_n \\ : \varepsilon_{n-1}(z) \varepsilon_{n-1}^*(z) : + : \varepsilon_n(z) \varepsilon_n^*(z) : & \text{for } D_n \end{cases}
\end{aligned}$$

Proof. First for type $X \neq C$ and $i, j = 1, \dots, n-1$ ($n-2$ for A_{n-1}), using Proposition 3.3 we have

$$\begin{aligned} & [X(\alpha_i, z), X(-\alpha_j, w)] \\ &= -[: \varepsilon_i(z) \varepsilon_{i+1}^*(z) :, : \varepsilon_{j+1}(w) \varepsilon_j^*(w) :] \\ &= -\delta_{ij} ([: \varepsilon_{i+1}^*(z) \varepsilon_{i+1}(z) : + : \varepsilon_i(z) \varepsilon_i^*(z) :] \delta(z-w) + \partial_w \delta(z-w)) \\ &= -\delta_{ij} (\alpha_i(z) \delta(z-w) + \partial_w \delta(z-w)) \end{aligned}$$

Similarly we have for $i, j = 1, \dots, n-1$ ($n-2$ for A_{n-1})

$$\begin{aligned} & [\alpha_i(z), \alpha_j(w)] \\ &= [: \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) :, : \varepsilon_j(z) \varepsilon_j^*(z) : - : \varepsilon_{j+1}(z) \varepsilon_{j+1}^*(z) :] \\ &= 2\delta_{ij} \partial_w \delta(z-w) - \delta_{i+1,j} \partial_w \delta(z-w) - \delta_{i,j+1} \partial_w \delta(z-w) \\ &= a_{ij} \partial_w \delta(z-w) = (\alpha_i | \alpha_j) \partial_w \delta(z-w), \end{aligned}$$

where a_{ij} are the entries of type A Cartan matrix.

More generally using Proposition 3.3 we have

$$\begin{aligned} & [X(\varepsilon_i - \varepsilon_j, z), X(\varepsilon_k - \varepsilon_l, w)] \\ &= \delta_{jk} X(\varepsilon_i - \varepsilon_l, z) \delta(z-w) - \delta_{li} X(\varepsilon_k - \varepsilon_j, z) \delta(z-w) \\ &\quad + \partial_w \delta(z-w). \end{aligned}$$

Now for any $k \neq l$ we have

$$\begin{aligned} & [\alpha_i(z), X(\varepsilon_k - \varepsilon_l, w)] \\ &= [: \varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_{i+1}(z) \varepsilon_{i+1}^*(z) :, : \varepsilon_k(w) \varepsilon_l^*(w)] \\ &= (\delta_{ik} - \delta_{il} - \delta_{i+1,k} + \delta_{i+1,l}) X(\varepsilon_k - \varepsilon_l, z) \delta(z-w) \\ &= (\alpha_i | \varepsilon_k - \varepsilon_l) X(\varepsilon_k - \varepsilon_l, z) \delta(z-w) \end{aligned}$$

It then follows that the operators $X(\alpha_i, z)$, $X(-\alpha_i, z)$ generate a type A subalgebra for $i = 1, \dots, n-1$ ($n-2$ for A_{n-1}).

In type C , similar computation gives for $1 \leq i, j \leq n-1$

$$[X(\alpha_i, z), X(-\alpha_j, w)] = -2\delta_{ij} (\alpha_i(z) \delta(z-w) + \partial_w \delta(z-w)),$$

and

$$\begin{aligned} & [\alpha_i(z), \alpha_j(w)] \\ &= d_i^{-1} d_j^{-1} 2(2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}) \partial_w \delta(z-w) \\ &= (\alpha_i | \alpha_j) \partial_w \delta(z-w), \end{aligned}$$

We now check for the special nodes. First for type A_{n-1} , if we start with $i = 0$ or $i = n-1$ the above relation is still valid provided we take

the indices modulo n and treating $\beta(z)$ as $\varepsilon_1(z)$. In particular for type A_{n-1} we have

$$\begin{aligned} & [X(\alpha_{n-1}, z), X(-\alpha_0, w)] \\ &= -[: \varepsilon_{n-1}(z) \varepsilon_0^*(z) :, : \varepsilon_n^*(w) \beta(w) :] \\ &= 0. \end{aligned}$$

$$\begin{aligned} & [X(\alpha_0, z), X(-\alpha_0, w)] = [: \varepsilon_n(z) \beta^*(z) :, : \varepsilon_n^*(w) \beta(w) :] \\ &= -(: \varepsilon_n^*(z) \varepsilon_n(z) : \delta(z-w) - : \beta(z) \beta^*(z) : \delta(z-w) - \partial_w \delta(z-w)) \\ &= -(\alpha_0(z) \delta(z-w) + \partial_w \delta(z-w)) \end{aligned}$$

Furthermore, in this case we have

$$\begin{aligned} & [\alpha_{n-1}(z), \alpha_0(w)] \\ &= [: \varepsilon_{n-1}(z) \varepsilon_{n-1}^*(z) : - : \varepsilon_n(z) \varepsilon_n^*(z) :, : \varepsilon_n(w) \varepsilon_n^*(w) : - : \beta(w) \beta^*(w) :] \\ &= -[: \varepsilon_n(z) \varepsilon_n^*(z) :, : \varepsilon_n(w) \varepsilon_n^*(w) :] \\ &= -\partial_w \delta(z-w) = (\alpha_{n-1} | \alpha_0) \partial_w \delta(z-w) \end{aligned}$$

In type B_n case we first have that for $k \neq l$

$$\begin{aligned} & [\alpha_0(z), X(\epsilon_k - \epsilon_l, w)] = -[: \beta(z) \beta^*(z) : + : \varepsilon_2(z) \varepsilon_2^*(z) :, : \varepsilon_k(w) \varepsilon_l^*(w) :] \\ &= (-\epsilon_1 - \epsilon_2 | \epsilon_k - \epsilon_l) X(\epsilon_k - \epsilon_l, w) \delta(z-w) \\ &= (\alpha_0 | \epsilon_k - \epsilon_l) X(\epsilon_k - \epsilon_l, w) \delta(z-w) \end{aligned}$$

$$\begin{aligned} & [X(\alpha_0, z), X(-\alpha_1, w)] = [: \beta^*(z) \varepsilon_2^*(z) :, : \varepsilon_1^*(w) \varepsilon_2(w) :] \\ &= - : \beta^*(z) \varepsilon_1^*(w) : \delta(z-w) = - : \varepsilon_1^*(z) \varepsilon_1^*(z) : \delta(z-w) = 0, \end{aligned}$$

where we have used the property that $\beta(z) = -\bar{c}(z) + \varepsilon_1(z)$ and $: \bar{c}(z) u(z) := 0$ for any $u \in \mathcal{C}$.

$$\begin{aligned} & [X(\alpha_0, z), X(-\alpha_0, w)] = [: \beta^*(z) \varepsilon_2^*(z) :, : \beta(w) \varepsilon_2(w) :] \\ &= -(: \beta^*(z) \beta(w) : + : \varepsilon_2^*(z) \varepsilon_2(w) :) \delta(z-w) - \partial_w \delta(z-w) \\ &= -(\alpha_0(z) \delta(z-w) + \partial_w \delta(z-w)) \end{aligned}$$

$$\begin{aligned} & [X(\alpha_{n-1}, z), X(-\alpha_n, w)] \\ &= [: \varepsilon_{n-1}(z) \varepsilon_n^*(z) :, : e(w) \varepsilon_n^*(w) :] = 0 \end{aligned}$$

In this case we also have

$$\begin{aligned}
 & [X(\alpha_n, z), X(-\alpha_n, w)] \\
 &= 2[: \varepsilon_n(z)e(z) :, : e(w)\varepsilon_n^*(w) :] \\
 &= -2[: \varepsilon_n(z)\varepsilon_n^*(w) : \delta(z-w) - : e(z)e(w) : \delta(z-w) + \partial_w \delta(z-w)] \\
 &= -2(\alpha_n(z)\delta(z-w) + \partial_w \delta(z-w)),
 \end{aligned}$$

where we have used the fact that $: e(z)e(z) := 0$.

Now for type D_n case the relations involving $i = 0$ are exactly same as in type B_n case. For the other relations we have:

$$\begin{aligned}
 & [X(\alpha_{n-1}, z), X(-\alpha_n, w)] \\
 &= [: \varepsilon_{n-1}(z)\varepsilon_n^*(z) :, : \varepsilon_{n-1}^*(w)\varepsilon_n^*(w) :] \\
 &= - : \varepsilon_n^*(z)\varepsilon_n^*(w) : \delta(z-w) \\
 &= - : \varepsilon_n^*(z)\varepsilon_n^*(z) : \delta(z-w) = 0
 \end{aligned}$$

$$\begin{aligned}
 & [X(\alpha_n, z), X(-\alpha_n, w)] \\
 &= [: \varepsilon_{n-1}(z)\varepsilon_n(z) :, : \varepsilon_{n-1}^*(w)\varepsilon_n^*(w) :] \\
 &= - ([: \varepsilon_n(z)\varepsilon_n^*(w) : + : \varepsilon_{n-1}(z)\varepsilon_{n-1}^*(w) :] \delta(z-w) + \partial_w \delta(z-w)) \\
 &= - (\alpha_n(z)\delta(z-w) + \partial_w \delta(z-w)),
 \end{aligned}$$

For type C_n we have

$$\begin{aligned}
 & [X(\alpha_0, z), X(-\alpha_0, w)] = \frac{1}{2}[: \beta^*(z)\varepsilon_1^*(z) :, : \beta(w)\varepsilon_1(w) :] \\
 & \quad + \frac{1}{2}[: \varepsilon_1^*(z)\bar{\beta}^*(z) :, : \varepsilon_1^*(w)\bar{\beta}^*(w) :] \\
 &= - (\alpha_0(z)\delta(z-w) + \partial_w \delta(z-w)).
 \end{aligned}$$

It is easy to see that $[X(\alpha_0, z), X(-\alpha_1, w)] = [X(\alpha_1, z), X(-\alpha_0, w)] = 0$. Moreover one has

$$\begin{aligned}
 & [X(\alpha_n, z), X(-\alpha_n, w)] = [: \varepsilon_n(z)\varepsilon_n^*(z) :, : \varepsilon_n^*(w)\varepsilon_n(w) :] \\
 & \quad - ([: \varepsilon_n^*(z)\varepsilon_n(z) : + : \varepsilon_n(z)\varepsilon_n^*(z) :] \delta(z-w) - \partial_w \delta(z-w)) \\
 &= - (\alpha_n(z)\delta(z-w) + \partial_w \delta(z-w)).
 \end{aligned}$$

As for the Serre relations, we first notice that it is easy to check that the OPE expansions of $X(\alpha_i, z)X(\alpha_i, w)$ or $X(-\alpha_i, z)X(-\alpha_i, w)$ are analytic for all $i = 0, \dots, n$, thus $[X(\pm\alpha_i, z), X(\pm\alpha_i, w)] = 0$.

For $i = j \pm 1$, we have:

$$\begin{aligned}
& [X(\alpha_i, z_1), [X(\alpha_i, z_2), X(\alpha_j, w)]] \\
&= [X(\alpha_i, z_1), (X(\alpha_i + \alpha_j, w)\delta_{i+1,j} - X(\alpha_j + \alpha_i, w)\delta_{j+1,i})\delta(z_2 - w) \\
&\quad + \delta_{i,j+1}\delta_{i+1,j}\partial_w\delta(z_2 - w)] \\
&= (X(2\alpha_i + \alpha_j, w)\delta_{i+1,i}\delta_{i,j+1} - X(\alpha_j + 2\alpha_i, w)\delta_{j+1,i}\delta_{j,i+1})\delta(z_1 - w) \\
&\quad \cdot \delta(z_2 - w) \\
&= 0.
\end{aligned}$$

Thus we have shown most cases and we include the verification for type B or D to show the method for the other vertices.

$$\begin{aligned}
& [X(\alpha_0, z_1), [X(\alpha_0, z_2), X(\alpha_2, w)]] \\
&= [X(\alpha_0, z_1), [:\beta^*(z_2)\varepsilon_2^*(z_2):, : \varepsilon_2(w)\varepsilon_3^*(w):]] \\
&= [:\beta^*(z_1)\varepsilon_2^*(z_1):, : \beta^*(z_2)\varepsilon_3^*(w):]\delta(z_2 - w) = 0.
\end{aligned}$$

$$\begin{aligned}
& [X(\alpha_{n-1}, z_1), [X(\alpha_{n-1}, z_2), X(\alpha_n, w)]] \\
&= [X(\alpha_{n-1}, z_1), [:\varepsilon_{n-1}(z_2)\varepsilon_n^*(z_2):, : \varepsilon_n(w)e(w):]] \\
&= [:\varepsilon_{n-1}(z_1)\varepsilon_n^*(z_1):, : \varepsilon_{n-1}(z_2)e(z_2):]\delta(z_2 - w) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
& [X(\alpha_n, z_1), [X(\alpha_n, z_2), [X(\alpha_n, z_3), X(\alpha_{n-1}, w)]]] \\
&= [X(\alpha_n, z_1), [X(\alpha_n, z_2), [:\varepsilon_n(z_3)e(z_3):, : \varepsilon_{n-1}(w)\varepsilon_n^*(w):]]] \\
&= [X(\alpha_n, z_1), [:\varepsilon_n(z_2)e(z_2):, : e(z_3)\varepsilon_{n-1}(z_3):]]\delta(z_3 - w) \\
&= -[X(\alpha_n, z_1), : \varepsilon_n(z_2)\varepsilon_{n-1}(z_2):]\delta(z_2 - z_3)\delta(z_3 - w) \\
&= -[: \varepsilon_n(z_1)e(z_1):, : \varepsilon_n(z_2)\varepsilon_{n-1}(z_2):]\delta(z_2 - z_3)\delta(z_3 - w) \\
&= 0
\end{aligned}$$

The remaining relations follow similarly. \square

REFERENCES

- [BB] S. Berman and Y. Billig, *Irreducible representations for toroidal Lie algebras*, J. of Algebra **221** (1999), 188-231.
- [EM] S. Eswara Rao and R. V. Moody, *Vertex representations for n -toroidal Lie algebras and a generalization of the Virasoro algebras*, Commun. Math. Phys. **159** (1994), 239-264.
- [F] I. B. Frenkel, *Spinor representations of affine Lie algebras*, Proc. Nat'l. Acad. Sci. USA, **77**, No. 11 (1980), 6303-6306.

- [FF] A. Feingold and I. B. Frenkel, *Classical affine algebras*, Adv. Math. **56** (1985), 117-172.
- [FJW] I. Frenkel, N. Jing, W. Wang, *Vertex representations via finite groups and the McKay correspondence* Int. Math. Res. Notices **4** (2000), 195-222.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.
- [FM] M. Fabbri and R. V. Moody, *Irreducible representations of Virasoro-toroidal Lie algebras*, Commun. Math. Phys. **159** (1994), 1-13.
- [G] Y. Gao, *Fermionic and bosonic representations of the extended affine Lie algebra $gl_N(\mathbb{C}_q)$* . Canad. Math. Bull. **45** (2002), no. 4, 623-633.
- [JMT] N. Jing, K. C. Misra, S. Tan, *Bosonic realizations of higher level toroidal Lie algebras*, Pacific J. Math., **219**, (2005), 285-302.
- [JMX] N. Jing, K. C. Misra, C. Xu, *Bosonic realizations of classical toroidal Lie algebras*, Proc. Amer. Math. Soc. to appear.
- [K] V. G. Kac, *Vertex algebras for beginners*, Univ. Lecture Ser., **10**, AMS, 1997.
- [L] M. Lau, *Bosonic and fermionic representations of Lie algebra central extensions*. Adv. Math. **194** (2005), no. 2, 225-245.
- [LH] D. Liu, N. Hu, *Vertex representations for toroidal Lie algebra of type G_2* , J. Pure Appl. Algebra **198** (2005), no. 1-3, 257-279.
- [MRY] R. V. Moody, S. E. Rao, T. Yokonuma, *Lie algebras and Weyl groups arising from vertex operator representations*, Nova J. Algebra Geom. **1** (1992), no. 1, 15-57.
- [T1] S. Tan, *Principal construction of the toroidal Lie algebra of type A_1* , Math. Zeit. **230** (1999), 621-657.
- [T2] S. Tan, *Vertex operator representations for toroidal Lie algebras of type B_1* , Commun. in Algebra **27** (1999), 3593-3618.
- [XH] L. Xia, N. Hu, *Irreducible representations for Virasoro-toroidal Lie algebras*, J. Pure Appl. Algebra **194** (2004), no. 1-2, 213-237.

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